2006 IEEE International Conference on
Multisensor Fusion and Integration for Intelligent Systems
September 3-6, 2006, Heidelberg, Germany

A Bayesian Approach to Information Fusion for Evaluating the Measurement Uncertainty

Klaus-Dieter Sommer, Olaf Kuehn, Fernando Puente Leòn, Bernd R. L. Siebert

Abstract - **The Bayesian approach to uncertainty evaluation is a classical example for information fusion. It is based on both, the knowledge about the measuring process and the input quantities. Appropriate probability density functions for the input quantities may be obtained by utilizing the principle of maximum information entropy and the Bayes theorem. The knowledge about the measurement process is represented by the so-called model equation which forms the basis for the fusion of all involved input quantities. Compared to the ISO-GUM procedure, the Bayesian approach to uncertainty evaluation does not have any restriction related to nonlinearity and determination of confidence intervals.**

I. Introduction

The concept which, in accordance with the "Guide to the Expression of Uncertainty in Measurement" [1] (here denoted as ISO-GUM), underlies the modern evaluation of measurement uncertainty is based on both, the available information about the measuring process and the (input) quantities and parameters that have influence on the value of the measurement result. The knowledge about the measuring process is to be condensed to the so-called model equation (fusion model). It mathematically represents the interrelation between the measurand Y and the involved input quantities $X_1, ..., X_N$ and their values respectively:

$$Y = f_{\mathrm{M}}\left(X_1, ..., X_N\right), \qquad (1a)$$

$$\eta = f_{\mathrm{M}}\left(\xi_1, ..., \xi_N\right). \qquad (1b)$$

Manuscript received April 15, 2006.

K.-D. Sommer is with the Thuringian State Bureau for Metrology and Verification (LMET), D-98693 Ilmenau, Germany (corresponding author to provide phone: +49(3677)850-101, fax: +49(3677)850-400; e-mail: klaus-dieter.sommer@lmet.de).

O. Kuehn is with the Thuringian State Bureau for Metrology and Verification (LMET), D-98693 Ilmenau, Germany (e-mail: olaf.kuehn@lmet.de).

F. Puente Leòn is with the Technische Universitaet Muenchen (Munich Technical University, TUM), Dep. of Electrical Engineering and Information Technology, D-80333 Munich, Germany (e-mail: f.puente@tum.de)

B. R. L. Siebert is with the Physikalisch-Technische Bundesanstalt (PTB), D-38116 Braunschweig, Germany (e-mail: bernd.sieber@ptb.de).

Therefore, the model equation might be understood as fusion model of the relevant input quantities.

Moreover, because it is presumed to represent the relationship between the input quantities and the measurand unique and completely, it additionally covers the scaling of the input quantities.

Any information about these input quantities is to be weighed as more or less relevant and reliable by assigning appropriate probability density functions (pdfs) $g_{Xi}\left(\xi_i\right)$ to them, where ξ_i are the possible values of X_i. In accordance with the Bayesian concept [2-6], a pdf represents the state of knowledge about the individual input quantity whence ever this knowledge is descended from.

The expectation value of a pdf, $x_i = \mathrm{E}\left[X_i\right]$, is taken for the best estimate of the quantity and the standard deviation of the pdf is taken as the standard uncertainty associated with the above expectation, $u_{xi} = \sqrt{\mathrm{Var}[X_i]}$.

II. Bayesian Description of Knowledge about Input Quantities

The state-of-knowledge pdf for any input quantity X_i may be obtained by utilizing the principle of maximum information entropy (pme) [2-3] that, for example, yields

- a rectangular pdf if one knows that the values ξ_i of the quantity X_i are contained in an interval (practical examples: given tolerances or error limits, digital resolution),
- a Gaussian (normal) pdf if one knows the best estimate x_j= E[X_j] and the associated standard uncertainty u_{xj} of the quantity X_j (practical examples: statement of a calibration result, result of a statistical analysis expressed by a mean and a standard deviation).

If new or additional information I_2, particularly in terms of measured data of the measurand, is available, the change of a (possibly) given prior pdf is described by the Bayes theorem [2-6]: The posterior pdf $g\left(\xi \middle| I_2, I_1\right)$ taking account of new data I_2 results from the prior information I_1 as

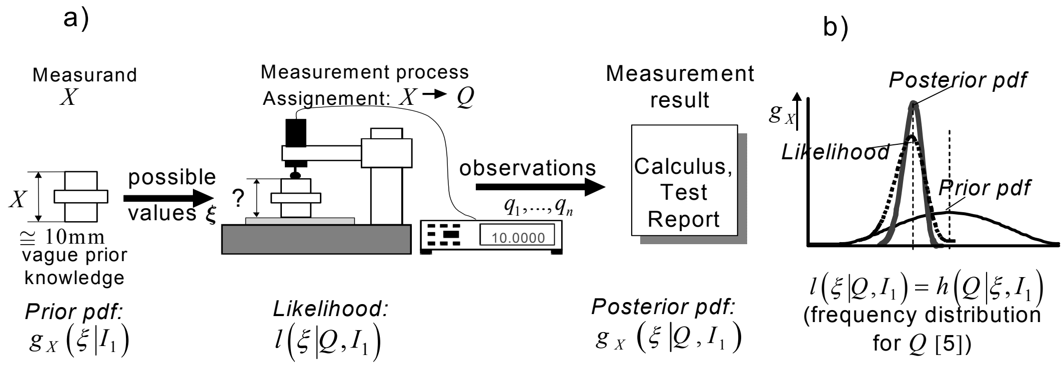

Fig. 1. Illustration of Bayesian inference in measurement: a) Measurement process.b) Probabilistic description of the state of knowledge. Symbols: $g\left(\xi|I_1\right)$ – prior pdf representing vague prior knowledge I_1 about the measurand, e.g. the nominal value and given error limits for the measurand; $l\left(\xi|Q,I_1\right)$ – likelihood representing the measuring process with the observed quantity Q ; $g\left(\xi|Q,I_1\right)$ – posterior pdf representing the available knowledge inferred from the likelihood and the prior pdf; $h\left(Q|\xi,I_1\right)$ – frequency distribution for the quantity Q

product of a normative constant *C*, the Likelihood $l\left(\xi|I_1,I_2\right)$ and the prior pdf:

$$g\left(\xi|I_2,I_1\right)d\xi = C\cdot l\left(\xi|I_2,I_1\right)\cdot g\left(\xi|I_1\right)d\xi. \quad (2)$$

For a repeatedly observed quantity Q which reflects the measurand *X*, the Bayesian approach is generally illustrated in Fig. 1. It should be noted that this way to uncertainty evaluation is not (yet) part of the ISO-GUM procedure [1] but the Joint Committee for Guides in Metrology of the Bureau International de Poids et Mesures (BIPM) is preparing further documents that are based consistently on the Bayesian probability theory [7].

III. Repeated Observations

Today, in practice, the prior knowledge about the measurand itself is usually neglected. In case of repeated observations $q_1,...,q_n$ of the quantity *Q*, a Gaussian probability model for any given datum q_k yields [3-4, 8]:

$$g(q_k|\varpi,\sigma_Q)\propto\sigma_Q^{-1}\cdot\exp\left[-\frac{\left(q_k-\varpi\right)^2}{2\sigma_Q}\right], \quad (3)$$

where ϖ represents the possible values of *Q*, and σ_Q are the possible values of the standard deviation associated with *Q*. The above pdf may be assumed to be equal to the frequency distribution for the observed data and is usually interpreted as being proportional to the Likelihood function $l\left(\varpi,\sigma_Q|Q\right)$ that is

$$l(\varpi,\sigma_Q|Q)\propto\sigma_Q^{-n}\cdot\exp\left[-\frac{\chi^2(\varpi,\sigma_Q,Q)}{2}\right] \quad (4)$$

where

$$\chi^2=\sum_{k=1}^{n}\left(\frac{q_k-\varpi}{\sigma_Q}\right)^2=\frac{n}{\sigma_Q^2}\left[\left(\varpi-\bar{q}\right)^2+\frac{n-1}{n}s_q^2\right],$$

$$s_q=\left(n-1\right)^{-1}\cdot\sum_{k=1}^{n}\left(q_k-\bar{q}\right)^2, \text{ and}$$

$$\bar{q}=n^{-1}\sum_{k=1}^{n}q_k .$$

By multiplying equation (4) with the non-informative Jeffrey's prior, the joint posterior pdf is obtained [3; 8]:

$$g\left(\varpi,\sigma_Q|Q\right)\propto\sigma_Q^{-(n+1)}\cdot\exp\left[-\frac{\chi^2\left(\varpi,\sigma_Q,Q\right)}{2}\right] \quad (5)$$

Integration to $d\sigma_Q$ leads to the information pertaining to the expectation for *Q*:

$$g\left(\varpi,\middle|Q\right) \propto \left\{1+\frac{\left[\left(\varpi-\bar{q}\right)/s\left(\bar{q}\right)\right]^2}{n-1}\right\}^{-n/2}, \tag{6}$$

where $s\left(\bar{q}\right)=s\left(q_k\right)\cdot n^{-1/2}$.

Since the right-hand side of this equation corresponds to a Student-t distribution, the new variable $T=\left(Q-\bar{q}\right)\cdot s^{-1}\left(\bar{q}\right)$ is introduced [8].

One obtains $g\left(t\right)\propto\left(1+\frac{t^2}{n-1}\right)^{-n/2}$,

where t are the possible Values of T. Therefore, the best estimate for Q, i.e. for the measurand, becomes

$$q=\mathrm{E}\left[Q\right]=\bar{q}=\frac{1}{n}\sum_{k=1}^{n}q_k \;. \tag{7}$$

Due to $\mathrm{Var}\left[T\right]=\left(n-1\right)\left(n-3\right)^{-1}$, the "Bayesian uncertainty contribution" associated with the expectation of the repeatedly observed quantity Q becomes [3, 8]

$$u_Q=\sqrt{\frac{n-1}{n\left(n-3\right)}}\cdot s\left(q_k\right) \;. \tag{8}$$

It should be mentioned that, for small numbers of observations $(n\leq 20)$, this uncertainty contribution significantly exceeds the so-called type-A uncertainty calculated in accordance with the ISO-GUM [1]. The GUM-type-A uncertainty, therefore, may be understood as approximation for a sufficiently large number of observations [10].

IV. Fusion of Information Given for the Input Quantities

It is the intrinsic purpose of the Bayesian approach to uncertainty analysis to develop the joint posterior pdf for the output quantity (measurand) which is compatible with the given information about the (values of the) input quantities and the measuring process.

$$g_Y\left(\eta\middle|Q,I\right)\propto l\left(\xi\middle|Q,I\right)\cdot g\left(\xi\middle|I\right), \tag{9}$$

where $g\left(\xi\right)$ represents the state of knowledge about the values of the input quantities $\xi=\xi_1,\ldots,\xi_N$.

Because of the interrelation of the input quantities (given by the equations (1a) and (1b)), *a posteriori* the input quantities cannot be acknowledged as being independent. This fact can be taken into consideration by writing the *joint prior* pdf as

$$g\left(\xi\middle|I\right)\propto g_X\left(\xi\right)\cdot g_{\mathrm{M}}\left(\xi\right) \text{ [4, 8]}, \tag{10}$$

where $g_X\left(\xi\right)$ is the prior of the input quantities and $g_{\mathrm{M}}\left(\xi\right)$ the so-called model prior [9]. The model equation itself may be obtained by systematically analyzing the cause-and-effect chain of the measurement including all relevant influences and disturbances [11-12].

For models of the form $M\left(X\right)=Y-f_{\mathrm{M}}\left(X\right)=0$, the above model prior is equal to Dirac's delta function [8]:

$$g_{\mathrm{M}}\left(\xi\right)=\delta\left[M\left(\eta\right)\right], \tag{11}$$

that "takes care" that only meaningful combinations of the possible values of the input quantities are taken into consideration ("filter function").

Therefore the joint posterior pdf for the output quantity becomes

$$g_Y\left(\eta\middle|Q,I\right)=\int_{\infty}^{\infty}\ldots\int_{\infty}^{\infty}g_{X1},\ldots,g_{XN}\left(\xi_1,\ldots,\xi_N\right)\cdot \delta\left(\eta-f_{\mathrm{M}}\left(\xi_1,\ldots,\xi_N\right)\right)d\xi_1,\ldots,d\xi_N \tag{12}$$

Equation (12) is known as *Markov formula*. Since it can be analytically computed in fairly simple cases only, modern uncertainty evaluation utilizes Monte-Carlo techniques as integration techniques for pdf propagation [13-15].

Fig. 2 illustrates the fully Bayesian concept for evaluating the measurement result and its associated uncertainty.

V. Expectation, Uncertainty and Expanded Uncertainty for the Output Quantity

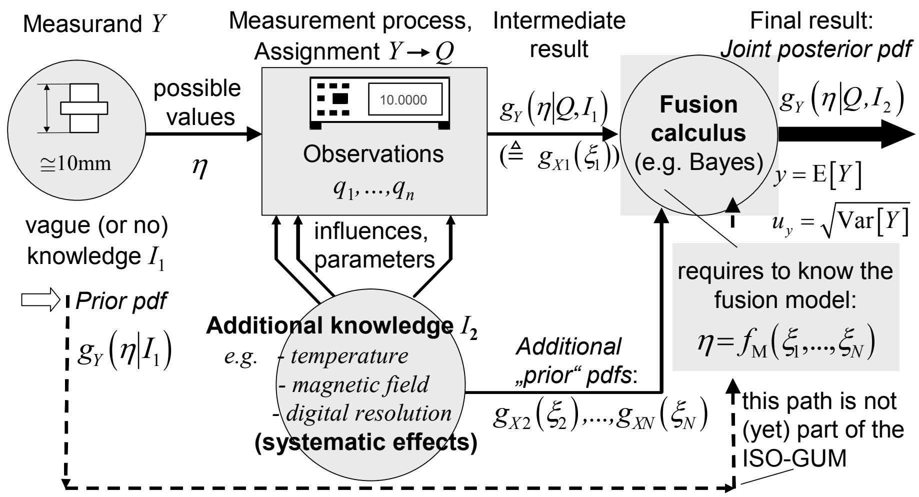

Fig. 2. Generalized measurement process as Bayesian inference by means of model-based fusion of the state-of-knowledge pdfs for the input quantities. Symbols: see text (depicted according to Beyerer [9])

From the pdf for the output quantity $g_Y(\eta|Q, I)$, the expectation value of the measurand $y = \mathrm{E}[Y]$ and its associated uncertainty u_y can be derived:

$$y = \int_{-\infty}^{+\infty} g_Y(\eta) \cdot \eta \, d\eta \text{, and} \tag{13}$$

$$u_y = \sqrt{\int_{-\infty}^{+\infty} g_Y(\eta)(\eta - y)^2 \, d\eta} \; . \tag{14}$$

Since the Bayesian approach does provide the pdf for the output quantity, the expanded uncertainty, i.e. a kind of confidence interval for this input quantity, can easily be derived as the minimum interval $[U_{P-}; U_{P+}]$ that meets the following coverage probability condition:

$$\int_{-\infty}^{U_{P+}} g_Y(\eta) d\eta - \int_{-\infty}^{U_{P-}} g_Y(\eta) d\eta = P \; . \tag{15}$$

Usually, this coverage probability P is set up to 0.95 at minimum [1].

VI. Linear Fusion Models (Linear Model Equations)

In practice, users of a measurement result will often not be interested in the pdf for the output quantity Y but rather in its expectation value y and the associated measurement uncertainty u_y (see equations (14) and (15)).

In case of linear systems or systems that can be linearized, e.g. by first-order Taylor series expansion, these parameters may also be computed in accordance with the ISO-GUM method [1] which is based on Gaussian uncertainty propagation: (see equations (13) and (14)).

$$y = f_{\mathrm{M}}(x_1, \ldots, x_N) \, , \tag{16}$$

$$u_y = \sqrt{\sum_{i=1}^{N} \left(\left. \frac{\partial f_{\mathrm{M}}}{\partial X_i} \right|_{xi} \right)^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left. \frac{\partial f_{\mathrm{M}}}{\partial X_i} \right|_{xi} \left. \frac{\partial f_{\mathrm{M}}}{\partial X_j} \right|_{xj} u_{xixj}} \tag{17}$$

where $y = \mathrm{E}[Y]$; $u_{xixj} = u_{xi} \cdot u_{xj} \cdot r(X_i; X_j)$ is the estimated covariance of the quantities X_i and X_j, and $r(X_i; X_j)$ is the respective correlation coefficient.

It is a common experience that, in the majority of practical uncertainty evaluations, the ISO-GUM procedure will provide satisfying results. But besides nonlinearity, the calculation of the expanded measurement uncertainty is a real weak point of the standard concept. The problem is caused by the fact that the standard procedure does not provide the pdf for the output quantity and, therefore, the coverage factor needed to calculate the expanded uncertainty is to be determined on the basis of only vague information about this pdf:

$$k_{\mathrm{P}} = U \cdot u_y \,. \tag{18}$$

VII. CONCLUSION

It becomes clear that, independent on the calculus used (Gaussian or Bayesian), for practitioners the key steps of modern uncertainty evaluation are the compilation and description of the knowledge about the measurement, the modelling of the measurement and the assignation of an appropriate pdf to each of the involved input quantities.

It can be concluded that the Bayesian approach allows for stringently evaluating the measurement uncertainty. There are no restrictions related to nonlinearity and determination of the expanded uncertainty. On the other hand, with the exceptions of evaluating the expanded uncertainty and calculating the standard uncertainty from only a few repeated observations, the ISO-GUM procedure is (for linearizable systems) consistent with the Bayesian concept (see also [10]).

REFERENCES

[1] Guide to the Expression of Uncertainty in Measurement (GUM), First edition, 1993, corrected and reprinted 1995, International Organization for Standardization (ISO), Geneva, 1993 and 1995.

[2] Weise, K.; Woeger, W.: A Bayesian theory of measurement uncertainty. Meas. Sci. Technol. 3(1992), 1-11

[3] Weise, K.; Woeger, W.: Messunsicherheit und Messdatenauswertung in der Metrologie. WILEY-VCH, Weinheim, 1999

[4] Lira, I.; Woeger, W.: Bayesian evaluation of the standard uncertainty and coverage probability in a simple measurement model. Meas. Sci. Technol. 12 (2001), 1172-1179

[5] Sivia, D.S.: Data Analysis – A Bayesian Tutorial. Clarendon Press, Oxford, 1996

[6] Estler, T.W.: Measurement as Inference: Fundamental Ideas. Annals of the CIRP 48 (1999) 2, 611–632

[7] Bich, W.; Cox, M.G.; Harris, P.M.: Evaluation of the "Guide to the Expression of Uncertainty in Measurement". Metrologia 43 (2006) 4

[8] Lira, I.: Evaluating the Measurement Uncertainty. Institute of Physics Publishing Bristol and Philadelphia Ltd, 2002

[9] Beyerer, J.: The value of additional knowledge in measurement – a Bayesian approach, Measurement 25 (1999) 1-7

[10] Kacker, R.; Jones, A.: On Use of Bayesian Statistics to Make the Guide to the Expression of Uncertainty in Measurement Consistent. Metrologia 40 (2003), 235-248

[11] Sommer, K.-D.; Siebert, B.R.L.; Kochsiek, M.; Weckenmann, A.: A systematic approach to the modelling of measurements for uncertainty evaluation. Journal of Physics: Conference Series 13 (2005), 7th International Symposium on Measurement Technology and Intelligent Instruments ISMTII 2005, Institute of Physics Publishing, 224-227

[12] Sommer, K.-D.; Siebert, B.R.L.: Systematic Approach the Modelling of Measurements for Uncertainty Evaluation. Metrologia 43 (2006) 4

[13] Cox, M.G.; Siebert, B.R.L.: The Use of Monte-Carlo Method for Evaluating Uncertainty and Expanded Uncertainty. Metrologia 43 (2006) 4

[14] Siebert, B.R.L.; Ciarlini, P.; Sibold, D.: Monte Carlo Study on Logical and Statistical Correlation in Advanced Mathematical & Computational Tools in Metrology V, edited by Ciarlini, P.; Filipe, E.; Forbes, AB.; Pavese, F.; Perruchet, C. and Siebert, B.R.L., Series on Advancesrd in Mathematics for applied Sciences, Vol. 57, World Scientific, New Jersey, 2006, S. 235-244.

[15] Siebert B.R.L.; Sommer, K.-D.: Weiterentwicklung des GUM und Monte-Carlo-Techniken / *New Developements of the GUM and Monte Carlo Techniques.* tm Technisches Messen 71 (2004), 2 (February 2004), 67 – 80 (ISSN 0178-2312)